# Response of a stochastic bistable model driven by strong time-dependent fields

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The long time response of a bistable stochastic model driven by large amplitude time-dependent sinusoidal fields is investigated by numerically solving the Langevin or the Fokker-Planck equation. The noise average behavior is oscillatory in time. For a given strength of the driving field, the behavior with the noise of the amplitude and phase of the fundamental harmonics of the average response depends on the frequency. For large driving frequencies, both magnitudes show maxima for two different values of the noise intensity. For a small enough external frequency, those maxima are absent.

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## I. INTRODUCTION

The dynamics of a nonlinear stochastic system, subject to the action of a time-dependent external force, show a variety of interesting phenomena which have been profusely studied in recent years [1,2]. In particular, a great deal of work has been devoted to the analysis of the long-time response of a stochastic bistable system driven by a sinusoidal time-dependent force. For weak enough fields, the long-time noise average of the relevant degree of freedom shows time oscillations with a frequency equal to the external one, $\Omega$, and out of phase with the driving term by an amount $-\phi_1$. The amplitude of the oscillations, $a_1$, and the phase-shift present nonmonotonic behaviors with the noise intensity when $\Omega$ is much smaller than the intrawell relaxation frequency. These behaviors have been analyzed by several analytical and numerical techniques [3]. The maximum in the phase shift reflects the competition between the interwell and intrawell motions, while the peak in the amplitude corresponds to the matching of the frequency of the noise induced switching events between wells and the driving frequency.

As long as the strength of the external force is kept sufficiently small, the bistable character is maintained during an external cycle. Then, the long-time probability distribution is only slightly distorted with respect to the unperturbed equilibrium one. It shows a bimodal structure with peak heights which oscillate in time. An analysis of the dynamics based on the ideas of linear-response theory (LRT) as carried out by Dykman *et al.* [3], seems adequate then. On the other hand, for strong fields, the bistable character of the potential is lost during an external period. The perturbation is so large that the validity of a LRT with an unperturbed spectral density appropriate for a bistable potential may become questionable. Jung and Hanggi [3] have recently studied the problem by numerically solving the corresponding Fokker-Planck equation (FPE) with a matrix continued fraction method. They find that, for some values of the external amplitude and frequency, the peak in the amplitude of the system response still survives, but the phase shift shows a monotonic behavior with $D$. We have also addressed this problem by numerically solving the FPE with a different numerical procedure [4]. Our results suggested that as the strength of the field is increased, both magnitudes should show peaks. The positions of the maxima are shifted with respect to the weak-field case with the peak of $-\phi_1$ showing up for a value of the noise intensity smaller than the one for which $a_1$ reaches its maximum. Unfortunately, the numerical procedure used in Ref. [4] is not reliable for very small values of the noise. Therefore, in this work we have explored this region of strong fields and very weak noises by numerically solving the corresponding Langevin equation. We find that both the amplitude and the phase shift of the system response show peaks for strong fields when the external frequency is large enough, while those peaks disappear for low-frequency fields.

The rest of the paper is organized as follows. In Sec. II, we analyze the dynamics of the system driven by a time-dependent force but in the absence of a noise term. The behavior of the deterministic trajectories is useful to understand the behavior when noise is present. The stochastic model is introduced in Sec. III. The results of numerical solutions of the evolution equations are discussed in Sec. IV. Finally, in Sec. V, we present our main conclusions.

## II. DYNAMICS IN THE ABSENCE OF NOISE

Let us consider a system with a relevant degree of freedom $x$, whose dynamical evolution is given by (in dimensionless form)

$$\frac{\partial x}{\partial t} = -U'(x) + 2S\cos\Omega t , \tag{1}$$

where the prime indicates the derivative of the bistable potential $U(x) = -x^2/2 + x^4/4$. $U(x)$ has two minima at the points $x = 1, -1$ separated by a barrier with height $\Delta U = \frac{1}{4}$ located at $x = 0$. The last term of Eq. (1) represents the effect of a driving external field.

In the case of zero frequency, ($\Omega = 0$), the external field breaks the symmetry of the potential. The location of the extrema and their heights depend upon the strength of the driving field in such a way that for $S > S_L(0) = \frac{1}{3}\sqrt{3}$, the potential loses its bistable character. For nonzero frequencies and small values of the amplitude of the driving field, $[S \ll S_L(0)]$, the system behavior can be understood by linearizing Eq. (1) around the minima of the wells. The deterministic trajectory remains confined within the region of attraction of the initial minima. For long times, the particle describes oscillations with frequency $\Omega$, amplitude $a = 2S/(4+\Omega^2)^{1/2}$ and out of phase with the driving field. The phase lag is given by $-\phi_0 = \arctan(\Omega/2)$. For frequencies much smaller than the well frequency, the amplitude $a$ is similar to that of the external term, and its value decreases as $\Omega$ is increased. The phase lag is very small for low frequencies and it tends to $-\pi/2$ for large ones. These results are corroborated by a numerical solution of Eq. (1) by means of a fourth-order Runge-Kutta integrator [5].

For strong fields, linearization around the minima is not correct and one has to resort to numerical solutions. The driving term distorts the shape of the unperturbed potential, $U(x)$, in such a way that the system might not feel a bistable potential during an entire external cycle and the trajectories may explore both regions of attraction depending upon the values of $\Omega$ and $S$. The long-time trajectories show oscillatory behavior. For a given $\Omega$, there exists a value of the field strength, $S_L(\Omega)$, such that for $S < S_L(\Omega)$, there are two centers of oscillations symmetrically located around $x=0$. For $S > S_L(\Omega)$ both centers coalesce into one located at zero. The transition points are indicated in Fig. 1. For very small frequencies, $S_L(\Omega)$ tends to its zero-frequency value $S_L(0)$. For parameter values corresponding to points lying below the circles, the trajectories might cross the origin but the centers of oscillations remain confined within the initial region of attraction although the strength of the driving can be substantially larger than the one required to destroy bistability in the static case. On the other hand, for parameter values above the transition line, the oscillations are around the origin with fairly large amplitude and out of phase with the driving term by a large amount. In this sense, one can say that the field alone is then capable of inducing switching between both regions of the $x$ axis.

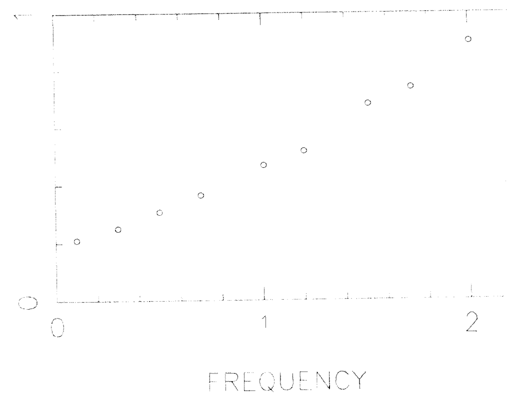


FIG. 1. Diagram showing the transition line in the $S,\Omega$ plane, characterizing the long-time deterministic trajectories (see text).

For low frequencies, the transition between both types of behavior is very abrupt. The two centers of oscillations remain located essentially around $\pm 1$ until $S$ reaches its transition value. Then their locations suddenly coalesce into zero. The amplitude of the oscillations also shows a sudden change in its value as $S$ crosses the transition line. This can be seen in the plots for $\Omega = 0.1$ in Fig. 2. On the other hand, for large frequencies, the oscillation centers gradually coalesce into zero as $S$ increases from $S_L(0)$ to its transition values $S_L(\Omega)$ and the amplitude has an almost linear behavior with $S$. An example of this is shown in Fig. 3 for $\Omega = 2.0$.

## III. STOCHASTIC SYSTEM

We now consider the effect of noise on the dynamics. The relevant degree of freedom satisfies now the

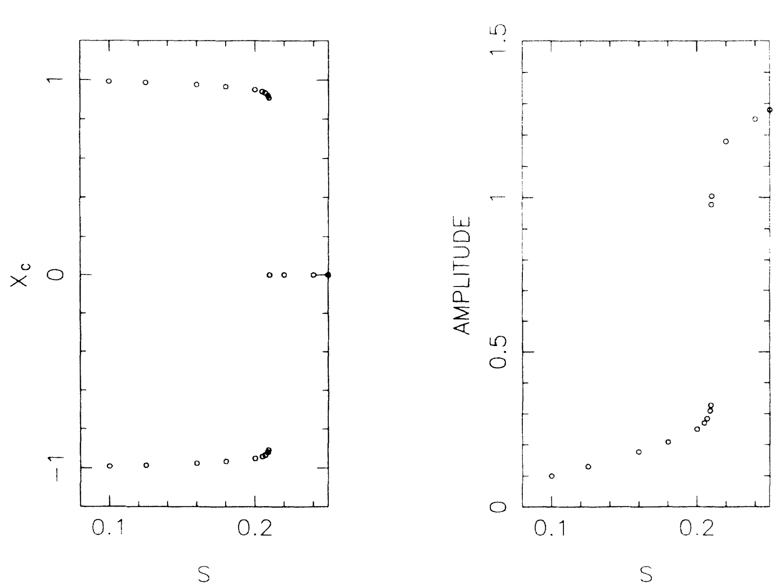


FIG. 2. Plots showing the dependence with $S$ of the oscillation centers ($X_c$) and the amplitude of the long-time deterministic trajectory for $\Omega = 0.1$.

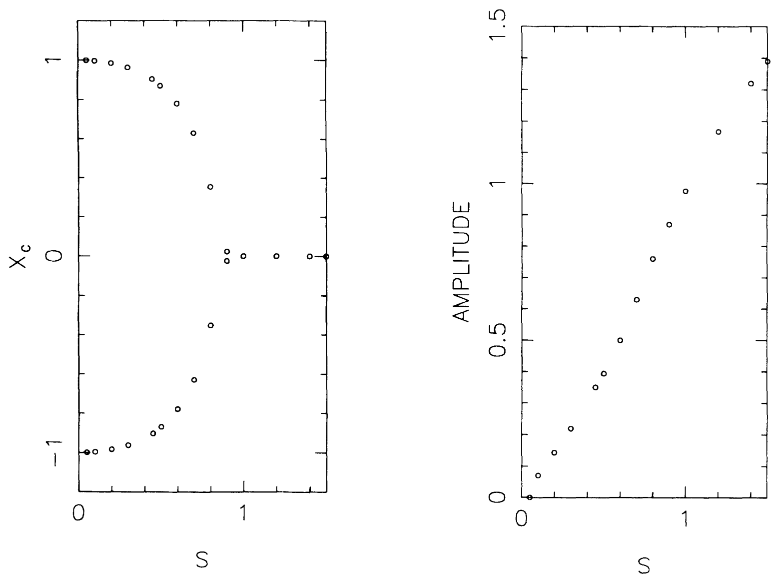


FIG. 3. Same as in Fig. 2, for $\Omega=2.0$.

Langevin equation

$$\frac{\partial x}{\partial t}=-U'(x)+2S\cos\Omega t+\eta(t) , \quad (2)$$

where $\eta(t)$ is a Gaussian white noise with zero mean and correlation function $\langle \eta(t)\eta(s)\rangle = D\delta(t-s)$. The corresponding FPE for the probability density is

$$\frac{\partial P(x,t)}{\partial t}=\frac{\partial}{\partial x}[U'(x)-2S\cos\Omega t]P+\frac{D}{2}\frac{\partial^2 P}{\partial x^2} . \quad (3)$$

As indicated in Sec. I, the properties of the nonstationary stochastic process $x(t)$ have been analyzed by a variety of analytical procedures which are reviewed in Refs. [1] and [2]. The long-time probability distribution is a time periodic function which depends only on the values of $D$, $S$, and $\Omega$ and is independent of the initial preparation of the system [6]. The long-time noise average $\langle x(t)\rangle_\infty$ shows oscillations described by the expression

$$\langle x(t)\rangle_\infty=\sum_{n=1} a_n\cos(n\Omega t+\phi_n) . \quad (4)$$

For small amplitude driving fields, it is possible to describe these oscillations by using, for instance, linear-response theory (LRT) [3]. As Dykman and co-workers have shown both the amplitude $a_1$ and the phase shift $\phi_1$ of the fundamental harmonic can be calculated in terms of the susceptibility of the system, which in turn is related with the spectral density of the unperturbed system via the fluctuation-dissipation theorem. The validity of LRT is restricted to situations such that the external field does not induce large deviations of the probability density with respect to its unperturbed form. But for strong driving fields, these deviations become so large that LRT might become invalid.

In this work, we are interested in the long-time response of the stochastic system driven by large amplitude fields. We will resort to the numerical solution of either the Langevin or the FP equation in order to analyze the combined effects of noise, nonlinearity, and external fields. For small values of the noise strength $D$, the Langevin equation is solved by generating a sufficiently large number of stochastic trajectories starting from a given initial condition $x(0)$ [7]. Averaging over the random trajectories one then finds the noise average behavior. For large $D$, this procedure becomes too expensive due to the excessive number of trajectories needed to get reliable statistics. Thus, for large $D$, we solve the corresponding FPE using a numerical technique based on the split operator method introduced by Feit, Fleck, and Steigel to deal with the time-dependent Schrödinger equation [8]. We have previously used this procedure in the analysis of the system in the absence of external field [9] as well as in the study of stochastic resonance for weak external fields [4]. The details of the numerical method can be found in Ref. [9].

## IV. RESULTS

Let us consider the long-time response of the system driven by a field with $S=0.2$ and $\Omega=0.1$. From the analysis of the deterministic limit ($D=0$) we know that the long-time solution corresponds to oscillations around a center near the minimum of the initial well and lagging behind the driving term by a small amount. For small values of $D$ ($D\leq 0.1$) the long-time noise average is obtained from the Langevin equation, while for larger values of $D$, the response is evaluated from the solution of the FPE. Fourier analysis of $\langle x(t)\rangle_\infty$ shows that the amplitude of the fundamental harmonics in the sum indicated in Eq. (4) is still much larger than those of the rest of the odd harmonics, while the even ones are zero as expected. For very small $D$, the amplitude $a_1$ and phase shift $-\phi_1$ of the oscillations are quite similar to those of the deterministic case, although the center of oscillation has now been shifted to the origin. This indicates that, even for very weak noise, sooner or later the particle is able to escape the initial well by contrast with the noise free case. As $D$ is increased, there is a sharp increase in both $a_1$ and $-\phi_1$, with the absolute value of the phase shift reaching a maximum at $D \approx 0.02$, a value about ten times smaller than that for which the amplitude reaches its maximum ($D \approx 0.24$). The fact that the peak in the phase shift shows up for smaller values of the noise than that of the amplitude was also present in the case of stochastic resonance with weak fields and low frequencies [3,4]. The overall nonmonotonic behavior observed in both magnitudes is presented in Fig. 4. Our results for the amplitude coincide with those reported by Jung and Hanggi [3], but are at variance with their findings for the phase shift.

For weak fields and $\Omega$ much smaller than the relaxation frequency within a well of the unperturbed potential, the analytical theories (LRT, two-state theory, etc.) show that the peak in the amplitude occurs for a value of $D$ such that the external frequency equals twice the Kramers frequency for the noise induced hopping over the potential barrier. For the strong field considered here, twice the Kramers frequency at the noise value for which $a_1$ has its maximum, is about 0.056 which is much smaller than the external frequency. To understand the behavior in Fig. 4 we notice that, even though the driving field is so strong that the time-dependent potential felt by the system loses its bistable character during an external period, the driving frequency is such that the deterministic trajectory remains confined within the initial attraction region. Thus the external field alone is not enough to induce the switching events. Once the noise is added, the particle will be able to cross the point $x=0$. The maximum observed indicates that there is a resonance effect between the driving frequency and the frequency of switching events between the wells of the unperturbed potential. This frequency is not due exclusively to the interplay of noise and nonlinearity but it is accelerated by the external field. It is then clear that the behavior observed for this strong field cannot be adequately described in terms of a LRT with a spectral density corresponding to an unperturbed bistable potential as in the weak-field case, where $D$ is the only parameter characterizing the hopping rate. Nonetheless, the qualitative feature of the presence of peaks in the amplitude and the phase shift are still present. The maximum in the phase shift arises because of the intrawell dynamics, while that of the amplitude reflects the matching of the external frequency with the hopping one. The output of the system indicates an amplification of the input signal for some range of values of $D$, but the degree of amplification is much smaller than in the weak-field case.

In Fig. 5 we show the effect of noise in the response of the system for $S=0.25$ and $\Omega=0.1$. From the analysis of Sec. II we know that for these parameter values, the external field alone is enough to induce switching between the wells. Then, monotonic behaviors with the noise strength in the amplitude and phase of the fundamental harmonics of the average response are expected. For very small $D$, the quantities $a_1$ and $-\phi_1$ have values which are close to their deterministic limits. As noise increases, $-\phi_1$ decreases monotonically, while the amplitude decreases first for $D<0.05$, then remains about constant until the noise strength is about 0.15, and it continues to decrease for higher values of $D$. For this small frequency, neither the amplitude nor the phase shift present maxima for finite values of $D$.

Let us now consider the behavior when the driving frequency is increased to $\Omega=2$, while $S$ is kept constant at $S=0.25$. As it is shown in Fig. 6, the phase $-\phi_1$ starts at its deterministic value, rising until $D \approx 0.4$. Then, it shows a broad maximum followed by a decrease towards values smaller than the deterministic one for large $D$. The amplitude shows a maximum for a large value of the noise, $D \approx 1.4$, and a secondary small peak for a small value of $D$ ($D \approx 0.12$) For these values of the parameters, the driving field alone is not enough to produce switching between the unperturbed wells. On the other hand, the primary maximum occurs for a value of $D$ so large that Kramers formula is not valid. In a previous work [9], we analyzed the relaxation process in the unperturbed model for large values of $D$ and we found a hopping frequency much larger than the one given by the

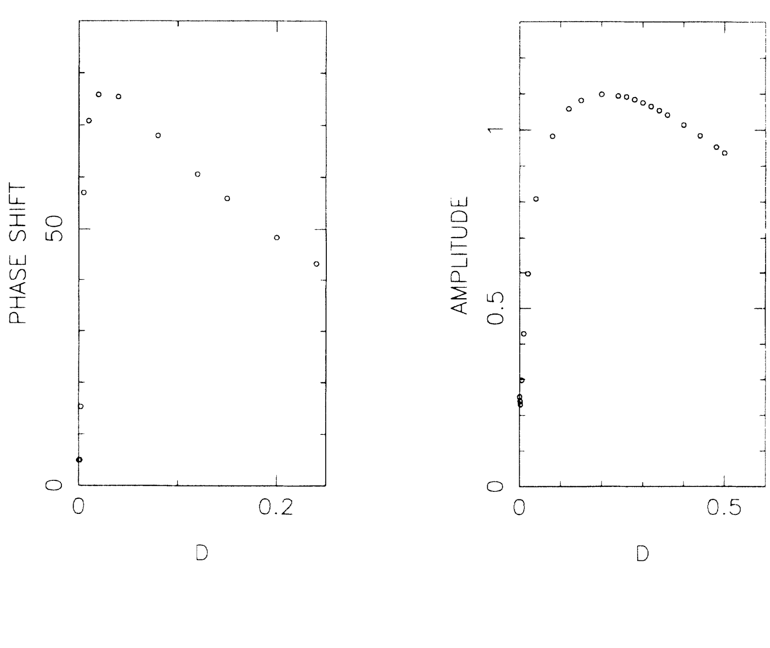


FIG. 4. Plot of the absolute value of the phase shift $\phi_1$ and the amplitude $a_1$ as functions of the noise strength $D$, for $S=0.2$ and $\Omega=0.1$.

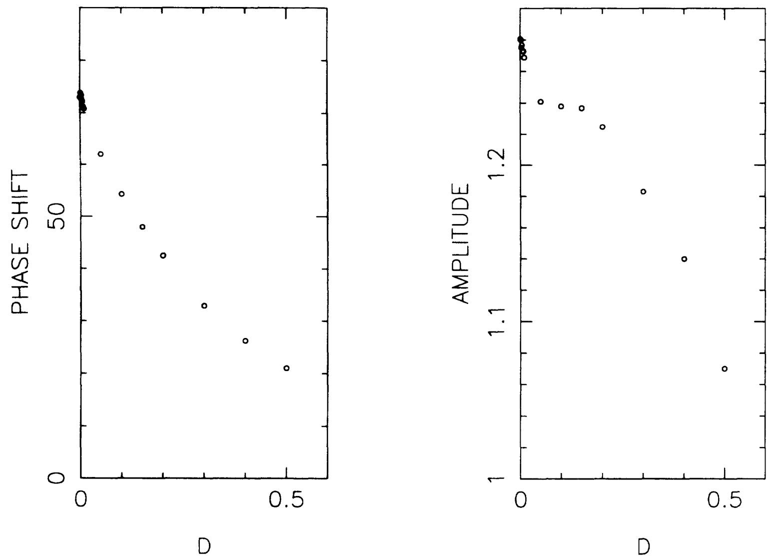


FIG. 5. Same as in Fig. 4, for $S=0.25$ and $\Omega=0.1$.

Kramers formula. Still, twice those values are much smaller than the external frequency considered here. The existence of the maximum then indicates that the hopping rate is accelerated by the external field with respect to its thermal value. It is worth noting that in this high-frequency case, the amplitude of the response is smaller than that of the driving term and there is no amplification by contrast with the low-frequency case of Fig. 4.

## V. CONCLUSIONS

In this work we have analyzed some aspects of the long-time response of a bistable stochastic system driven by large amplitude external fields. In the absence of noise, the particle trajectory may or may not cross the origin depending upon the values of the field parameters, even though its amplitude is larger than the one required to destroy bistability at zero frequency. The numerical solutions of the Langevin or the FP equations indicate that the amplitude and phase shift of the fundamental harmonics of the response show monotonic behaviors with the noise strength when the external field alone induces switching between the wells. On the other hand, if the driving parameters are such that noise is needed to observe switching, then the amplitude and phase shift of the fundamental harmonics of the long-time response of the system show maxima for some values of $D$. As in the weak-field case analyzed in Refs. [3] and [4], the maximum of the phase lag appears for a lower value of the noise than that for which the amplitude is maximum. The mechanisms giving rise to those maxima are essentially the same as for weak fields. Namely, the behavior of the phase lag shows the effect of the intrawell dynamics, while the maxima of the amplitude reflects the matching of twice the hopping frequency and the external one. By contrast with the low-field case, the maxima appears at such values of $D$ for which the hopping rate is not well described by Kramers formula. Large amplitude fields accelerate the hopping rate with respect to its

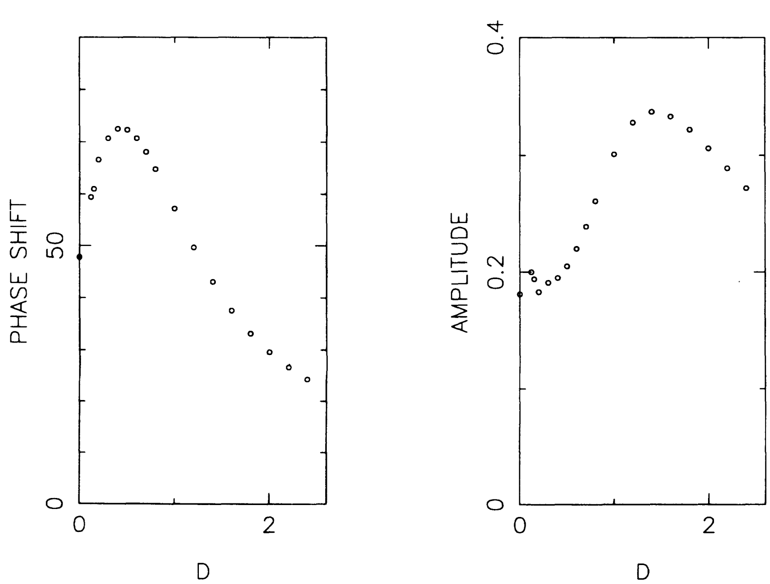


FIG. 6. Same as in Fig. 4, for $S=0.25$ and $\Omega=2.0$.

thermal value. The response of the system shows a small amplification with respect to the input term when the driving frequency is small. For large frequencies, even though the amplitude shows a maximum, it is very flat and there is no amplification.

## ACKNOWLEDGMENTS

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